

# UP-TO-ONE APPROXIMATIONS OF SECTIONAL CATEGORY AND TOPOLOGICAL COMPLEXITY

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**ABSTRACT.** James' sectional category and Farber's topological complexity are studied in a general and unified framework.

We introduce 'relative' and 'strong relative' forms of the category for a map and show that both can differ from sectional category by just one. A map has sectional or relative category at most  $n$  if, and only if, it is 'dominated' in a (different) sense by a map with strong relative category at most  $n$ . A homotopy pushout can increase sectional category but neither homotopy pushouts, nor homotopy pullbacks, can increase (strong) relative category. This makes (strong) relative category a convenient tool to study sectional category. We completely determine the sectional and relative categories of the fibres of the Ganea fibrations.

In particular, the 'topological complexity' of a space is the sectional category of the diagonal map, and so it can differ from the (strong) relative category of the diagonal by just one. We call the strong relative category of the diagonal 'strong complexity'. We show that the strong complexity of a suspension is at most two.

Our aim is to build a general and unified framework to study James' *sectional category* of a map [10], and Farber's *topological complexity* of a space, which is the sectional category of the diagonal map [7].

In the first section, we give definitions of *secat* in the 'Ganea-Whitehead style', and introduce variants of *secat*, called *relative category* and *strong relative category*. The main result of this section, Theorem 18, is that all these variants differ by just one. Also there is a kind of 'attachment formula' for relative category, which is Proposition 22. These results, and others, come essentially from Lemmas 11 and 14 which assert that homotopy pushouts and homotopy pullbacks do not increase (strong) relative category. It is fruitful to jointly consider sectional and relative categories, because they do not share the same properties: see Lemma 11 or Proposition 29 for instance. At the end of the section, we show in Corollary 33 that the sectional category of the fibre  $F_i \rightarrow G_i$  of the  $i^{\text{th}}$  Ganea fibration  $G_i \rightarrow X$  is  $\min\{i, \text{cat}(X)\}$ , while its relative category is  $\min\{i + 1, \text{cat}(X)\}$ . Actually, this result comes as a particular case of the determination of the sectional category and relative category of maps of some 'relative' Ganea construction, which is given in Theorem 32.

In the second section, we apply the results of the first section to complexity. Variants of complexity, corresponding to variants of sectional category, differ by just one. In particular, the strong relative category of the diagonal  $\Delta: X \rightarrow X \times X$  is called the *strong complexity* of  $X$ , and in Theorem 43 we show, by an explicit computation, that the strong complexity of any suspension is at most two.

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We work in the category of well-pointed topological spaces  $\mathbf{Top}^{\mathbf{w}}$  (*well-pointed* means that the inclusion of the base point is a closed cofibration) [14]. But we don't use any construction particular to topological spaces, we use only homotopy pullbacks and homotopy pushouts; so our techniques also apply in algebraic categories used to model topological spaces (commutative differential graded algebras, modules over a d.g.a., etc.). More precisely, our results apply in the general context of a closed model category  $\mathcal{M}$ , satisfying the *Cube axiom* (see the appendix for details). However, in these categories, in order to use the usual property of 'homotopy', one needs cofibrant-fibrant objects, but it's possible to circumvent this difficulty with some technical complications; see [4] or [3].

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## 1. SECTIONAL CATEGORY

### 1.1. The Ganea construction.

**Definition 1.** For any map  $\iota_X : A \rightarrow X$  of  $\mathcal{M}$ , the *Ganea construction* of  $\iota_X$  is the following sequence of homotopy commutative diagrams ( $i \geq 0$ ):

$$\begin{array}{ccccc}
 & & A & & \\
 & \nearrow \eta_i & & \searrow \alpha_{i+1} & \\
 F_i & & & & G_{i+1} \xrightarrow{g_{i+1}} X \\
 & \searrow \beta_i & & \nearrow \gamma_i & \\
 & & G_i & & 
 \end{array}$$

(Note: The diagram above is a simplified representation of the one in the image. The actual diagram in the image shows a homotopy pullback square with vertices  $F_i$ ,  $A$ ,  $G_i$ , and  $G_{i+1}$ , and a homotopy pushout square with vertices  $G_i$ ,  $G_{i+1}$ , and  $X$ . The maps are  $\eta_i: F_i \rightarrow A$ ,  $\alpha_{i+1}: A \rightarrow G_{i+1}$ ,  $\beta_i: F_i \rightarrow G_i$ ,  $\gamma_i: G_i \rightarrow G_{i+1}$ ,  $g_{i+1}: G_{i+1} \rightarrow X$ , and  $g_i: G_i \rightarrow X$ . The map  $\iota_X: A \rightarrow X$  is also shown as a diagonal arrow from  $A$  to  $X$ .)

where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map  $g_{i+1}: G_{i+1} \rightarrow X$  is the whisker map induced by this homotopy pushout. The induction starts with  $g_0 = \iota_X: A \rightarrow X$ .

We denote  $G_i$  by  $G_i(\iota_X)$ , or by  $G_i(X, A)$ . If  $\mathcal{M}$  is pointed with  $*$  as zero object, we write  $G_i(X) = G_i(X, *)$ .

The sequence of homotopy commutative diagrams above extends to:

$$\begin{array}{ccccc}
 & & A & & \\
 & \nearrow \alpha_i & & \searrow \alpha_{i+1} & \\
 A \xleftarrow{\theta_i} F_i & & & & G_{i+1} \xrightarrow{g_{i+1}} X \\
 & \searrow \beta_i & & \nearrow \gamma_i & \\
 & & G_i & & 
 \end{array}$$

(Note: The diagram above is a simplified representation of the one in the image. The actual diagram in the image shows a homotopy pullback square with vertices  $A$ ,  $F_i$ ,  $G_i$ , and  $G_{i+1}$ , and a homotopy pushout square with vertices  $G_i$ ,  $G_{i+1}$ , and  $X$ . The maps are  $\alpha_i: A \rightarrow F_i$ ,  $\alpha_{i+1}: A \rightarrow G_{i+1}$ ,  $\beta_i: F_i \rightarrow G_i$ ,  $\gamma_i: G_i \rightarrow G_{i+1}$ ,  $g_{i+1}: G_{i+1} \rightarrow X$ , and  $g_i: G_i \rightarrow X$ . The map  $\iota_X: A \rightarrow X$  is also shown as a diagonal arrow from  $A$  to  $X$ . The map  $\theta_i: A \rightarrow F_i$  is the whisker map induced by the homotopy pullback  $F_i$ .)

where  $\alpha_0 = \text{id}_A$  and, since  $g_i \circ \alpha_i \simeq \iota_X$ , the map  $\theta_i: A \rightarrow F_i$  is the whisker map induced by the homotopy pullback  $F_i$ . Notice also that  $\gamma_i \circ \alpha_i \simeq \alpha_{i+1}$ .

**Definition 2.** Let  $\iota_X: A \rightarrow X$  be a map of  $\mathcal{M}$ .

1) The *sectional category* of  $\iota_X$  is the least integer  $n$  such that the map  $g_n: G_n(\iota_X) \rightarrow X$  has a homotopy section, i.e. there exists a map  $\sigma: X \rightarrow G_n(\iota_X)$  such that  $g_n \circ \sigma \simeq \text{id}_X$ .

2) The *relative category* of  $\iota_X$  is the least integer  $n$  such that the map  $g_n: G_n(\iota_X) \rightarrow X$  has a homotopy section  $\sigma$  and  $\sigma \circ \iota_X \simeq \alpha_n$ .

We denote the sectional category by  $\text{secat}(\iota_X)$  or  $\text{secat}(X, A)$ , and the relative category by  $\text{relcat}(\iota_X)$  or  $\text{relcat}(X, A)$ . If  $\mathcal{M}$  is pointed with  $*$  as zero object, we write  $\text{cat}(X) = \text{secat}(X, *) = \text{relcat}(X, *)$ .

In **Top<sup>w</sup>**,  $\text{cat}(X)$  is T. Ganea's version of the *category* of L. Lusternik and L. Schnirelmann. For a normal path-connected space  $X$  with non-degenerate base-point, the definition here is equivalent to the original one with open covers of  $X$  (up to a shift by 1):  $\text{LS-cat}(X) = \text{cat}(X) + 1$ .

For a comprehensive review on category and sectional category of topological spaces and maps, see [10]; these notions are also deeply analysed in [2]. Warning: Our relative category is *not* that of E. Fadell [6].

**1.2. Strong pushout category.** Here we define the ‘strong pushout category’ of a map and establish its basic properties. The principal result here is Proposition 6, which characterizes sectional and relative category in terms of this invariant.

**Definition 3.** The *(strong) pushout category* of a map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$  is the least integer  $n$  such that:

- There are maps  $\iota_0: A \rightarrow X_0$  and a homotopy inverse  $\lambda: X_0 \rightarrow A$ , i.e.  $\iota_0 \circ \lambda \simeq \text{id}_{X_0}$  and  $\lambda \circ \iota_0 \simeq \text{id}_A$ ;
- for each  $i$ ,  $0 \leq i < n$ , there exists a homotopy pushout

$$\begin{array}{ccc} Z_i & \xrightarrow{\quad} & A \\ \downarrow & & \downarrow \iota_{i+1} \\ X_i & \xrightarrow{\chi_i} & X_{i+1} \end{array}$$

such that  $\chi_i \circ \iota_i \simeq \iota_{i+1}$ ;

- $X_n = X$  and  $\iota_n \simeq \iota_X$ .

We denote the strong pushout category by  $\text{Pushcat}(\iota_X)$ , or  $\text{Pushcat}(X, A)$ .

In particular,  $\text{Pushcat}(\iota_X) = 0$  iff  $\iota_X$  is a homotopy equivalence. When this is not true, then  $\text{Pushcat}(\iota_X) = 1$  if there is a homotopy pushout:

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & A \\ \rho' \downarrow & & \downarrow \iota_X \\ A & \xrightarrow{\iota_X} & X \end{array}$$

(The maps  $\rho$  and  $\rho'$  are *not* necessarily homotopic, unless  $\iota_X$  is a homotopy monomorphism.)

For  $0 \leq i < n$ , define the sequence of maps  $\xi_i = \xi_{i+1} \circ \chi_i$  beginning with  $\xi_n = \text{id}_X$ . We have  $\xi_n \circ \iota_n \simeq \iota_X$  and  $\xi_i \circ \iota_i = \xi_{i+1} \circ \chi_i \circ \iota_i \simeq \xi_{i+1} \circ \iota_{i+1} \simeq \iota_X$  by decreasing induction. So we have a sequence of homotopy commutative diagrams:

$$\begin{array}{ccccc} & & A & & \\ & \nearrow & & \searrow \iota_X & \\ Z_i & & & & X \\ & \searrow & & \nearrow \iota_X & \\ & & X_{i+1} & \xrightarrow{\xi_{i+1}} & X \\ & \nearrow \chi_i & & \nearrow \xi_i & \\ & X_i & & & \end{array}$$

where the inside square is a homotopy pushout and the map  $\xi_{i+1}$  is the whisker map induced by this homotopy pushout. Notice that  $\xi_0 \simeq \xi_0 \circ \iota_0 \circ \lambda \simeq \iota_X \circ \lambda$ .

Observe that, clearly,  $\text{Pushcat}(\iota_i) \leq i$ . Also we have:

**Lemma 4.** *Let  $\iota_X: A \rightarrow X$  be a map of  $\mathcal{M}$ . Suppose we have a homotopy pushout:*

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & A \\ \downarrow & & \downarrow \iota_C \\ X & \xrightarrow{\chi} & C \end{array}$$

where  $\chi \circ \iota_X \simeq \iota_C$ . Then, we have  $\text{Pushcat}(\iota_C) \leq \text{Pushcat}(\iota_X) + 1$ .

**Definition 5.** Consider the following diagram

$$\begin{array}{ccc} & & X \\ & \nearrow \iota_X & \uparrow \sigma \\ A & & \\ & \searrow \iota_Y & \downarrow \varphi \\ & & Y \end{array}$$

such that  $\varphi \circ \iota_X \simeq \iota_Y$ .

1) If  $\sigma$  is a homotopy section of  $\varphi$ , i.e.  $\varphi \circ \sigma \simeq \text{id}_Y$ , we say that  $\iota_Y$  is *(simply) dominated by  $\iota_X$* .

2) If  $\sigma$  is a homotopy section of  $\varphi$  and  $\sigma \circ \iota_Y \simeq \iota_X$ , we say that  $\iota_Y$  is *relatively dominated by  $\iota_X$* .

3) If  $\sigma$  is a homotopy inverse of  $\varphi$ , i.e.  $\varphi \circ \sigma \simeq \text{id}_Y$  and  $\sigma \circ \varphi \simeq \text{id}_X$ , we say that  $\iota_Y$  and  $\iota_X$  are of *the same homotopy type*. Notice that in this case, we have also  $\sigma \circ \iota_Y \simeq \iota_X$ .

Our definitions of  $\text{secat}(\iota_X)$ ,  $\text{relcat}(\iota_X)$ ,  $\text{Pushcat}(\iota_X)$ , and  $\text{Relcat}(\iota_X)$  below, are designed in such a way that they become *de facto* invariants of homotopy type.

**Proposition 6.** *Let  $\iota_Y: A \rightarrow Y$  be a map of  $\mathcal{M}$ . The following conditions are equivalent:*

- (i)  $\text{secat}(\iota_Y) \leq n$  (respectively:  $\text{relcat}(\iota_Y) \leq n$ );
- (ii) the map  $\iota_Y$  is simply (respectively: relatively) dominated by a map  $\iota_X: A \rightarrow X$  (see Definition 5 above) such that  $\text{Pushcat}(\iota_X) \leq n$ .

*Proof.* Consider the map  $\alpha_n: A \rightarrow G_n(\iota_Y)$  as in Definition 1 and notice that  $\text{Pushcat}(\alpha_n) \leq n$ . If  $\text{secat}(\iota_Y) \leq n$ , then  $\iota_Y$  is simply dominated by  $\alpha_n$ . If  $\text{relcat}(\iota_Y) \leq n$ , then  $\iota_Y$  is relatively dominated by  $\alpha_n$ .

For the reverse direction, we suppose the existence of maps as in Definition 5 such that  $\varphi \circ \iota_X \simeq \iota_Y$  and  $\varphi \circ \sigma \simeq \text{id}_Y$ . From  $\text{Pushcat}(\iota_X) \leq n$ , we get a sequence of homotopy commutative diagrams, for  $0 \leq i < n$ , as in Definition 3, which gives the top part of the following diagram.

We show by induction that the map  $\varphi \circ \xi_i: X_i \rightarrow Y$  factors through  $g_i: G_i(\iota_Y) \rightarrow Y$  up to homotopy. This is true for  $i = 0$  since we have  $\xi_0 \simeq \iota_X \circ \lambda$ , so  $\varphi \circ \xi_0 \simeq \varphi \circ \iota_X \circ \lambda \simeq \iota_Y \circ \lambda = g_0 \circ \lambda$ . Suppose now that we have a map  $\lambda_i: X_i \rightarrow G_i(\iota_Y)$

such that  $g_i \circ \lambda_i \simeq \varphi \circ \xi_i$ . Then we construct a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & Z_i & \xrightarrow{\quad} & A \\
 & \swarrow & \vdots & \nearrow \iota_{i+1} & \parallel \\
 X_i & \xrightarrow{\quad} & X_{i+1} & \xrightarrow{\xi_{i+1}} & X \\
 \downarrow \lambda_i & & \downarrow \lambda_{i+1} & & \downarrow \varphi \\
 & & F_i & \xrightarrow{\quad} & A \\
 & \swarrow & \vdots & \nearrow \alpha_{i+1} & \parallel \\
 G_i(\iota_Y) & \xrightarrow{\quad} & G_{i+1}(\iota_Y) & \xrightarrow{g_{i+1}} & Y
 \end{array}$$

where  $Z_i \dashrightarrow F_i$  is the whisker map induced by the bottom homotopy pullback and  $\lambda_{i+1}: X_{i+1} \dashrightarrow G_{i+1}(\iota_Y)$  is the whisker map induced by the top homotopy pushout. The composite  $g_{i+1} \circ \lambda_{i+1}$  is homotopic to  $\varphi \circ \xi_{i+1}$  by Lemma 49. Hence the inductive step is proven.

At the end of the induction, we have  $g_n \circ \lambda_n \simeq \varphi \circ \xi_n = \varphi \circ \text{id}_X = \varphi$ . As we have a homotopy section  $\sigma: Y \rightarrow X_n = X$  of  $\varphi$ , we get a homotopy section  $\lambda_n \circ \sigma$  of  $g_n$ . If, in addition,  $\sigma \circ \iota_Y \simeq \iota_X$ , then  $\lambda_n \circ \sigma \circ \iota_Y \simeq \lambda_n \circ \iota_X \simeq \lambda_n \circ \iota_n \simeq \alpha_n$ .  $\square$

**Corollary 7.** *Let  $\iota_Y: A \rightarrow Y$  be simply (respectively: relatively) dominated by  $\iota_X: A \rightarrow X$ . Then we have  $\text{secat}(\iota_Y) \leq \text{secat}(\iota_X)$  (respectively:  $\text{relcat}(\iota_Y) \leq \text{relcat}(\iota_X)$ ).*

**1.3. Strong relative category.** Here we define the ‘strong relative category’ of a map and establish its basic properties. The principal results here are Lemmas 11 and 14 which assert that homotopy pushouts and homotopy pullbacks do not increase (strong) relative category.

**Definition 8.** The *strong relative category* of a map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$  is the least integer  $n$  such that:

- There are maps  $\iota_0: A \rightarrow X_0$  and a homotopy inverse  $\lambda: X_0 \rightarrow A$ , i.e.  $\iota_0 \circ \lambda \simeq \text{id}_{X_0}$  and  $\lambda \circ \iota_0 \simeq \text{id}_A$ ;
- for each  $i$ ,  $0 \leq i < n$ , there exists a homotopy commutative diagram

$$\begin{array}{ccccc}
 & & & & A \\
 & & & \nearrow \rho_i & \searrow \iota_{i+1} \\
 A & \dashrightarrow \sigma_i & Z_i & & X_{i+1} \\
 & \searrow \iota_i & & \nearrow \chi_i & \\
 & & X_i & & 
 \end{array}$$

where the inside square is a homotopy pushout;

- $X_n = X$  and  $\iota_n \simeq \iota_X$ .

We denote the strong relative category by  $\text{Relcat}(\iota_X)$ , or  $\text{Relcat}(X, A)$ .

If  $\mathcal{M}$  is pointed with  $*$  as zero object, we write  $\text{Cat}(X) = \text{Relcat}(X, *) = \text{Pushcat}(X, *)$ .

In  $\mathbf{Top}^w$ ,  $\text{Cat}(X)$  is the homotopy invariant version of the R. Fox’s *strong category*, see [8].

In particular,  $\text{Relcat}(\iota_X) = 0$  iff  $\iota_X$  is a homotopy equivalence. When this is not true, then  $\text{Relcat}(\iota_X) = 1$  if there is a homotopy pushout:

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & A \\ \rho' \downarrow & & \downarrow \iota_X \\ A & \xrightarrow{\iota_X} & X \end{array}$$

such that  $\rho$  and  $\rho'$  have a common homotopy section  $\sigma$ . The following proposition shows that this situation occurs for any homotopy cofibre:

**Proposition 9.** *Assume  $\mathcal{M}$  is pointed. Given a homotopy cofibration sequence  $Y \rightrightarrows A \xrightarrow{\iota_X} X$ , we have  $\text{Relcat}(\iota_X) \leq 1$ .*

*Proof.* To get a homotopy pushout as above, use Proposition 50 and let  $Z = A \vee Y$ ,  $\rho \simeq (\text{id}_A, f)$  is the whisker map of  $\text{id}_A$  and  $f$ , and  $\rho' \simeq \text{pr}_1: A \vee Y \rightarrow A$  and  $\sigma \simeq \text{in}_1: A \rightarrow A \vee Y$  are the obvious maps.  $\square$

In section 2.2, we will construct a two-steps example showing that the diagonal  $\Delta: \Sigma X \rightarrow \Sigma X \times \Sigma X$  of a suspension has  $\text{Relcat}(\Delta) \leq 2$ .

Observe that for the map  $\alpha_i: A \rightarrow G_i(\iota_X)$  of the Ganea construction, we have  $\text{Relcat}(\alpha_i) \leq i$ . And since, clearly,  $\text{Pushcat}(\iota_X) \leq \text{Relcat}(\iota_X)$ , Proposition 6 can be extended to the following:

**Proposition 10.** *Let  $\iota_Y: A \rightarrow Y$  be a map of  $\mathcal{M}$ . The following conditions are equivalent:*

- (i)  $\text{secat}(\iota_Y) \leq n$  (respectively:  $\text{relcat}(\iota_Y) \leq n$ );
- (ii) the map  $\iota_Y$  is simply (respectively: relatively) dominated by a map  $\iota_X: A \rightarrow X$  such that  $\text{Pushcat}(\iota_X) \leq n$ ;
- (iii) the map  $\iota_Y$  is simply (respectively: relatively) dominated by a map  $\iota_{X'}: A \rightarrow X'$  such that  $\text{Relcat}(\iota_{X'}) \leq n$ .

**Lemma 11.** *If  $\iota_X: A \rightarrow X$  and  $A \rightarrow B$  are maps of  $\mathcal{M}$ , consider the homotopy pushout:*

$$\begin{array}{ccc} A & \xrightarrow{\iota_X} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\kappa_S} & S \end{array}$$

*We have  $\text{Relcat}(\kappa_S) \leq \text{Relcat}(\iota_X)$  and  $\text{relcat}(\kappa_S) \leq \text{relcat}(\iota_X)$ .*

*Proof.* Let  $\text{Relcat}(\iota_X) = n$ . Consider the sequence of homotopy pushouts as in Definition 8 and the sequence of maps  $\xi_i$  defined after Definition 3. This gives the top part of the next diagram. We extend it to a sequence of homotopy commutative

diagrams, for  $0 \leq i < n$ ,

$$\begin{array}{ccccccc}
 A & \xrightarrow{\sigma_i} & Z_i & \xrightarrow{\rho_i} & A & & \\
 \downarrow \iota_i & & \downarrow & & \downarrow \iota_{i+1} & & \downarrow \iota_X \\
 & & X_i & \xrightarrow{\quad} & X_{i+1} & \xrightarrow{\xi_{i+1}} & X \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B & \xrightarrow{\tilde{\sigma}_i} & R_i & \xrightarrow{\tilde{\rho}_i} & B & & \\
 \downarrow \tilde{\iota}_i & & \downarrow & & \downarrow \tilde{\iota}_{i+1} & & \downarrow \kappa_S \\
 & & S_i & \xrightarrow{\quad} & S_{i+1} & \xrightarrow{\quad} & S
 \end{array}$$

building all vertical faces as homotopy pushouts. From the Prism lemma 46, we know that the bottom face of the inside cube is a homotopy pushout and that  $\tilde{\rho}_i \circ \tilde{\sigma}_i \simeq \text{id}_B$ . Also, since  $\iota_0: A \rightarrow X_0$  has a homotopy inverse,  $\tilde{\iota}_0: B \rightarrow S_0$  has a homotopy inverse as well. Finally, since  $\xi_n = \text{id}_X$ , we may assume  $S_n = S$  and  $\tilde{\iota}_n \simeq \kappa_S$ . This means  $\text{Relcat}(\kappa_S) \leq n$ .

Now let  $\text{relcat}(\iota_X) = n$ . We can build the same diagrams, except that  $\xi_n$  has only a homotopy section  $\sigma$  such that  $\sigma \circ \iota_X \simeq \iota_n$ . By the Prism lemma 46,  $S_n \rightarrow S$  has a homotopy section  $\tilde{\sigma}$  such that  $\tilde{\sigma} \circ \kappa_S \simeq \tilde{\iota}_n$ . By Proposition 10, this means  $\text{relcat}(\kappa_S) \leq n$ .  $\square$

Observe that it is *not* true that  $\text{secat}(\kappa_S) \leq \text{secat}(\iota_X)$ . For instance, if  $\mathcal{M}$  is pointed with  $*$  as zero object, choose  $A$  so that its suspension is not contractible, and choose  $X = B = *$ , hence  $S = \Sigma A$ ; then  $\text{secat}(\iota_*) = 0$  while  $\text{secat}(\kappa_{\Sigma A}) = \text{cat}(\Sigma A) = 1$ .

**Corollary 12.** *Assume  $\mathcal{M}$  is pointed, and let  $A \rightarrow X \rightarrow C$  be a homotopy cofibration. Then  $\text{cat}(C) \leq \text{relcat}(X, A)$  and  $\text{Cat}(C) \leq \text{Relcat}(X, A)$ .*

**Corollary 13.** *Assume that  $\iota_X: A \rightarrow X$  is a ‘homotopy retract’ of  $\kappa_Y: B \rightarrow Y$ , i.e. there exists a homotopy commutative diagram in  $\mathcal{M}$ :*

$$\begin{array}{ccccc}
 A & \xrightarrow{t} & B & \xrightarrow{\zeta} & A \\
 \downarrow \iota_X & & \downarrow \kappa_Y & & \downarrow \iota_X \\
 X & \xrightarrow{s} & Y & \xrightarrow{f} & X
 \end{array}$$

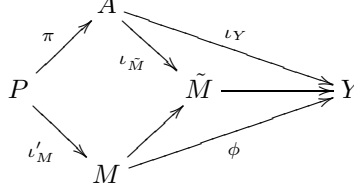
such that  $f \circ s \simeq \text{id}_X$  and  $\zeta \circ t \simeq \text{id}_A$ . Then  $\text{relcat}(\iota_X) \leq \text{relcat}(\kappa_Y)$ .

*Proof.* Consider the homotopy pushout  $S$  of  $\kappa_Y$  and  $\zeta$ :

$$\begin{array}{ccc}
 B & \xrightarrow{\zeta} & A \\
 \downarrow \kappa_Y & & \downarrow \iota_S \\
 Y & \xrightarrow{q} & S
 \end{array}$$

and let  $j: S \rightarrow X$  be the whisker map of  $f$  and  $\iota_X$ . We have  $j \circ q \circ s \simeq f \circ s \simeq \text{id}_X$ , so  $q \circ s$  is a homotopy section of  $j$ . Also we have  $q \circ s \circ \iota_X \simeq q \circ \kappa_Y \circ t \simeq \iota_S \circ \zeta \circ t \simeq \iota_S$ . This means that  $\iota_X$  is relatively dominated by  $\iota_S$ , and we obtain  $\text{relcat}(\iota_X) \leq \text{relcat}(\iota_S)$  by Proposition 7. But we also know that  $\text{relcat}(\iota_S) \leq \text{relcat}(\kappa_Y)$  by Lemma 11. Hence  $\text{relcat}(\iota_X) \leq \text{relcat}(\kappa_Y)$ .  $\square$

**Lemma 14.** *If  $\iota_Y: A \rightarrow Y$  and  $\phi: M \rightarrow Y$  are maps of  $\mathcal{M}$ , consider the following join construction:*



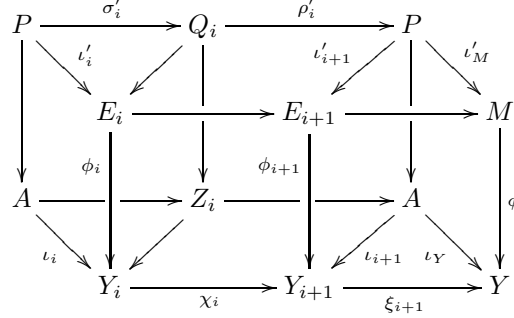
where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map  $\tilde{M} \rightarrow Y$  is the whisker map induced by the homotopy pushout. We have

$$\text{Relcat}(\iota_{\tilde{M}}) \leq \text{Relcat}(\iota'_M) \leq \text{Relcat}(\iota_Y)$$

and

$$\text{relcat}(\iota_{\tilde{M}}) \leq \text{relcat}(\iota'_M) \leq \text{relcat}(\iota_Y).$$

*Proof.* Let  $\text{Relcat}(\iota_X) = n$ . Consider the sequence of homotopy pushouts as in Definition 8 and the sequence of maps  $\xi_i$  defined after Definition 3. This gives the bottom part of the next diagram. We extend it to a sequence of homotopy commutative diagrams, for  $0 \leq i < n$ ,



building all vertical faces as homotopy pullbacks; since  $\xi_n = \text{id}_X$ , we may assume  $\phi_n = \phi$ . From the Cube axiom 45, we know that the top face of the inside cube is a homotopy pushout. Notice that  $\sigma'_i$  is a homotopy section of  $\rho'_i$ ; also, since  $\iota_0$  has a homotopy inverse,  $\iota'_0$  has a homotopy inverse, too; and finally  $\iota'_n: P \rightarrow E_n$  is  $\iota'_M: P \rightarrow M$ . This means  $\text{Relcat}(\iota'_M) \leq n$ .

Now let  $\text{relcat}(\iota_Y) = n$ . We can build the same diagrams, except that  $\xi_n$  has only a homotopy section  $\sigma$  such that  $\sigma \circ \iota_Y \simeq \iota_n$ . By the Prism lemma 46, the map  $E_n \rightarrow M$  has a homotopy section  $\sigma'$  such that  $\sigma' \circ \iota'_M \simeq \iota'_n$ . By Proposition 10, this means  $\text{relcat}(\iota'_M) \leq n$ .

The remaining inequalities are direct applications of Lemma 11.  $\square$

Notice the two interesting particular cases:

**Corollary 15.** *Assume  $\mathcal{M}$  is pointed. Let  $f: F \rightarrow E$  be the homotopy fibre of  $E \rightarrow B$ . Then  $\text{cat}(E/F) \leq \text{relcat}(f) \leq \text{cat}(B)$  and  $\text{Cat}(E/F) \leq \text{Relcat}(f) \leq \text{Cat}(B)$ .*

**Corollary 16.** *Consider the Ganea construction of any map  $\iota_X: A \rightarrow X$  made in Definition 1. For any  $i \geq 0$ , we have  $\text{relcat}(\alpha_{i+1}) \leq \text{relcat}(\beta_i) \leq \text{relcat}(\iota_X)$ .*

We end this subsection by observing that, clearly:



**Lemma 17.** *Suppose we have a homotopy commutative diagram where the square is a homotopy pushout:*

$$\begin{array}{ccccc} A & \xrightarrow{\sigma} & Z & \xrightarrow{\rho} & A \\ & \searrow \iota_X & \downarrow \tau & & \downarrow \iota_C \\ & & X & \xrightarrow{\chi} & C \end{array}$$

and where  $\rho \circ \sigma \simeq \text{id}_A$ . Then, we have  $\text{Relcat}(\iota_C) \leq \text{Relcat}(\iota_X) + 1$ .

**1.4. Comparing all these invariants.** Notice that, for any map  $\iota_X: A \rightarrow X$ , Proposition 6 implies  $\text{relcat}(\iota_X) \leq \text{Pushcat}(\iota_X)$ . On the other hand, we have obvious inequalities:  $\text{secat}(\iota_X) \leq \text{relcat}(\iota_X)$  and  $\text{Pushcat}(\iota_X) \leq \text{Relcat}(\iota_X)$ . One might think that these four integers could be quite different; indeed, for instance,  $\text{secat}(\iota_X) = 0$  iff  $\iota_X$  has a homotopy section, while  $\text{relcat}(\iota_X) = 0$  iff  $\iota_X$  is a homotopy equivalence. But in fact the four integers can differ only by 1, as is shown by the following result, which is an enhancement of a classical result of F. Takens [15].

**Theorem 18.** *For any map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$ , we have:*

$$\text{secat}(\iota_X) \leq \text{relcat}(\iota_X) \leq \text{Pushcat}(\iota_X) \leq \text{Relcat}(\iota_X) \leq \text{secat}(\iota_X) + 1.$$

*Proof.* We have just observed the first three inequalities.

Let  $\text{secat}(\iota_X) = n$  and let  $\sigma: X \rightarrow G_n$  be a homotopy section of the Ganea map  $g_n: G_n \rightarrow X$ . Use  $\sigma$  as  $\phi$  and  $\alpha_n: A \rightarrow G_n$  as  $\iota_Y$  in Lemma 14 to get  $\tilde{X} = X \vee_P A$  with  $\text{Relcat}(\iota_{\tilde{X}}) \leq \text{Relcat}(\alpha_n) \leq n$ .

By definition of  $P$ , we have a homotopy commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\iota'_X} & X \\ \pi \downarrow & & \downarrow \sigma \\ A & \xrightarrow{\alpha_n} & G_n \xrightarrow{g_n} X \\ & \searrow \iota_X & \nearrow \end{array}$$

therefore the map  $\iota'_X: P \rightarrow X$  factors through  $\iota_X: A \rightarrow X$  up to homotopy. This factorization allows the construction of the following homotopy commutative diagram where each square is a homotopy pushout:

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & A & & \\ \pi \downarrow & & \downarrow b & \searrow & \\ A & \xrightarrow{a} & \tilde{A} & \xrightarrow{j} & A \\ \iota_X \downarrow & & \downarrow f & & \downarrow \iota_X \\ X & \xrightarrow{s} & \tilde{X} & \xrightarrow{r} & X \end{array}$$

We have  $\iota_X \circ \pi \simeq \iota'_X$ , so  $f \circ b \simeq \iota_{\tilde{X}}$  by the Prism lemma 46. The map  $j$  is the whisker map of two copies of  $\text{id}_A$  induced by the homotopy pushout  $\tilde{A}$ ; so  $j \circ b \simeq \text{id}_A$ . As a consequence of Lemma 17, we have  $\text{Relcat}(\iota_X) \leq \text{Relcat}(\iota_{\tilde{X}}) + 1 \leq n + 1$ .  $\square$

We recover Proposition 9 as a corollary of Theorem 18:

**Corollary 19.** *Assume  $\mathcal{M}$  is pointed. Given a homotopy cofibration sequence  $A \rightarrow B \rightarrow C$ , we have  $\text{Relcat}(C, B) \leq 1$ .*

*Proof.* Since the map  $\iota_*: A \rightarrow *$  to the zero object has a homotopy section  $* \rightarrow A$ , we have  $\text{secat}(\iota_*) = 0$ . So Theorem 18 gives  $\text{Relcat}(\iota_*) \leq 1$ , and Lemma 11 gives the result.  $\square$

The following corollary shows that the sectional and relative categories of a map differ whenever the category of its homotopy cofibre is greater than the category of its target:

**Corollary 20.** *Assume  $\mathcal{M}$  is pointed. For any map  $\iota_X: A \rightarrow X$  with homotopy cofibre  $C$  such that  $\text{cat}(X) < \text{cat}(C)$ , we have  $\text{secat}(\iota_X) = \text{cat}(X)$  and  $\text{relcat}(\iota_X) = \text{cat}(C) = \text{cat}(X) + 1$ .*

*Proof.* By Proposition 29, we have  $\text{secat}(\iota_X) \leq \text{cat}(X)$  and by Corollary 12, we have  $\text{cat}(C) \leq \text{relcat}(\iota_X)$ . So, by the hypothesis,  $\text{secat}(\iota_X)$  and  $\text{relcat}(\iota_X)$  must differ at least by 1. On the other hand, by Theorem 18,  $\text{secat}(\iota_X)$  and  $\text{relcat}(\iota_X)$  can differ at most by 1. Hence we obtain the desired equalities.  $\square$

**Example 21.** The homotopy cofibre of the Hopf fibration  $h: S^3 \rightarrow S^2$  is  $\mathbb{C}P^2$  and we have  $\text{cat}(S^2) = 1 < \text{cat}(\mathbb{C}P^2) = 2$ . Thus  $\text{secat}(h) = 1$  and  $\text{relcat}(h) = 2$ .

In particular, Corollary 20 shows that the category  $\text{cat}(C)$  of the homotopy cofibre  $C$  of any map  $A \rightarrow X$  is always less than or equal to  $\text{cat}(X) + 1$  (a well-known result in the context of topological spaces). We extend this with the following proposition:

**Proposition 22.** *With the same notations and hypotheses as in Lemma 4, we have*

$$\text{relcat}(\iota_C) \leq \text{relcat}(\iota_X) + 1.$$

*Proof.* Suppose  $\text{relcat}(\iota_X) \leq n$  and consider the section  $\sigma: X \rightarrow G_n(\iota_X)$  of the map  $g_n: G_n(\iota_X) \rightarrow X$  given by Definition 2. Construct  $\tilde{X}$  with  $\text{Relcat}(\tilde{X}, A) \leq n$  as in the proof of Theorem 18. Since  $\sigma \circ \iota_X \simeq \alpha_n$  we get a whisker map  $A \rightarrow P$ :

$$\begin{array}{ccccc} & & \iota_X & & \\ & \curvearrowright & & \curvearrowright & \\ A & \cdots \rightarrow & P & \rightarrow & X \\ & \searrow & \downarrow \pi & & \downarrow \sigma \\ & & A & \xrightarrow{\alpha_n} & G_n \end{array}$$

thus  $\pi$  is a homotopy epimorphism, which implies that  $a \simeq b$ ; therefore  $\iota_{\tilde{X}} \simeq f \circ b \simeq f \circ a \simeq s \circ \iota_X$ . Now consider the following homotopy pushouts

$$\begin{array}{ccccccc} & & A & \xrightarrow{a} & \tilde{A} & \xrightarrow{j} & A \\ & & \downarrow \iota_X & & \downarrow f & & \downarrow \iota_X \\ Z & \xrightarrow{\tau} & X & \xrightarrow{s} & \tilde{X} & \xrightarrow{r} & X \\ \downarrow \rho & & \downarrow \chi & & \downarrow \tilde{\chi} & & \downarrow \chi \\ A & \xrightarrow{\iota_C} & C & \xrightarrow{s'} & \tilde{C} & \xrightarrow{r'} & C \end{array} \quad \begin{array}{c} \curvearrowright \\ \iota_C \end{array}$$

Recall that  $j \circ a \simeq \text{id}_A$ , so  $r \circ s \simeq \text{id}_X$  and  $r' \circ s' \simeq \text{id}_C$ ; we deduce that  $\iota_C: A \rightarrow C$  is relatively dominated by  $s' \circ \iota_C: A \rightarrow \tilde{C}$ . On the other hand, we have a homotopy pushout

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & A \\ s \downarrow \tau & & \downarrow s' \iota_C \\ \tilde{X} & \xrightarrow{\tilde{\chi}} & \tilde{C} \end{array}$$

Moreover  $\tilde{\chi} \circ \iota_{\tilde{X}} \simeq \tilde{\chi} \circ s \circ \iota_X \simeq s' \circ \chi \circ \iota_X \simeq s' \circ \iota_C$ ; so by Lemma 4 we see that  $\text{Pushcat}(\tilde{C}, A) \leq n + 1$ . Finally we deduce from Proposition 6 that  $\text{relcat}(C, A) \leq n + 1$ .  $\square$

We already know that  $\text{relcat}(G_i(\iota_X), A)$  is less than or equal to both  $i$  and  $\text{relcat}(X, A)$ . In fact, one can make this relation more precise. The following is an extension of a result of O. Cornea [1].

**Proposition 23.** *Let  $\iota_X: A \rightarrow X$  be any map of  $\mathcal{M}$ . Consider the map  $\alpha_i: A \rightarrow G_i(\iota_X)$  of the Ganea construction. We have:*

$$\text{relcat}(\alpha_i) = \min\{i, \text{relcat}(\iota_X)\}.$$

*Proof.* Let  $\text{relcat}(X, A) = n$ .

Suppose  $i \geq n$ . The map  $g_n: G_n(\iota_X) \rightarrow X$  has a homotopy section  $\sigma$  such that  $\sigma \circ \iota_X \simeq \alpha_n$ , therefore  $g_i: G_i(\iota_X) \rightarrow X$  has a homotopy section  $\sigma_i = \gamma_{i-1} \circ \dots \circ \gamma_n \circ \sigma$  and  $\sigma_i \circ \iota_X \simeq \alpha_i$ . So we have  $\text{relcat}(X, A) \leq \text{relcat}(G_i(\iota_X), A)$  from Corollary 7. On the other hand, Corollary 16 gives us the reverse inequality, and so the equality is proved.

Now suppose  $i < n$ , and assume that  $\text{relcat}(G_i(\iota_X), A) \leq i - 1$ . From Proposition 22, we deduce that  $\text{relcat}(G_{i+1}(\iota_X), A) \leq i$ , and so on, until we obtain  $\text{relcat}(G_n(\iota_X), A) \leq n - 1$ . This contradicts  $\text{relcat}(G_n(\iota_X), A) \geq n$  established above, so we have  $\text{relcat}(G_i(\iota_X), A) > i - 1$ . On the other hand, clearly, we have  $\text{relcat}(G_i(\iota_X), A) \leq i$ , and the equality is proved.  $\square$

**1.5. The Whitehead construction.** In this subsection, we assume that the category  $\mathcal{M}$  is pointed.

**Definition 24.** For any map  $\iota_X: A \rightarrow X \in \mathcal{M}$ , the *Whitehead construction* or *fat wedges* of  $\iota_X$  is the following sequence of homotopy commutative diagrams ( $i > 0$ ):

$$\begin{array}{ccccc} & & X^i \times A & & \\ & \nearrow t_{i-1} \times \text{id}_A & \searrow \text{id}_{X^i} \times \iota_X & & \\ T_{i-1} \times A & & & & T_i \xrightarrow{t_i} X^{i+1} \\ & \searrow \text{id}_{T_{i-1}} \times \iota_X & \nearrow v_i & & \\ & & T_{i-1} \times X & \xrightarrow{t_{i-1} \times \text{id}_X} & \end{array}$$

where the outside square is a homotopy pullback, the inside square is a homotopy pushout, and the map  $t_i: T_i \rightarrow X^{i+1}$  is the whisker map induced by this homotopy pushout. The induction starts with  $t_0 = \iota_X: A \rightarrow X$ .

We denote  $T_i$  by  $T_i(\iota_X)$ , or by  $T_i(X, A)$ . We also write  $T_i(X) = T_i(X, *)$ .

The following result is (almost) Theorem 8 in [9] :

**Theorem 25.** *Let  $\iota_X: A \rightarrow X$  be a map of  $\mathcal{M}$ . Let  $i \geq 0$  and consider the diagonal map  $\Delta_{i+1}: X \rightarrow X^{i+1}$ . Then we have homotopy pullbacks:*

$$\begin{array}{ccccc} A & \xrightarrow{\alpha_i} & G_i(\iota_X) & \xrightarrow{g_i} & X \\ \delta_i \downarrow & & \downarrow \varepsilon_i & & \downarrow \Delta_{i+1} \\ X^i \times A & \xrightarrow{v_i} & T_i(\iota_X) & \xrightarrow{t_i} & X^{i+1} \end{array}$$

*Proof.* We proceed inductively. Consider the following homotopy commutative diagram, where  $\delta_i$  is the whisker map of  $\Delta_i \circ \iota_X$  and  $\text{id}_A$ , and  $\bar{\epsilon}_{i-1}$  is the whisker map of  $\epsilon_{i-1}$  and  $g_{i-1}$ :

$$\begin{array}{ccccc} A & \xrightarrow{\iota_X} & X & \xleftarrow{g_{i-1}} & G_{i-1} \\ \delta_i \downarrow & & \downarrow \Delta_{i+1} & & \downarrow \bar{\epsilon}_{i-1} \\ X^i \times A & \xrightarrow{\quad} & X^i \times X & \xleftarrow{\quad} & T_{i-1} \times X \\ \swarrow & & \swarrow & & \swarrow \\ A & \xrightarrow{\iota_X} & X & & X^i \xleftarrow{t_{i-1}} T_{i-1} \end{array}$$

Applying the Prism lemma 46 to the left and right parts of the diagram to obtain that the two upper squares are homotopy pullbacks. Now apply the Join theorem 51 to the two upper squares to get the inductive step.  $\square$

We denote  $\epsilon_i$  by  $\epsilon_i(\iota_X)$  or  $\epsilon_i(X, A)$ ; we also write  $\epsilon_i(X) = \epsilon_i(X, *)$ . Notice that  $\delta_i(\text{id}_X) \simeq \epsilon_i(\text{id}_X) \simeq \Delta_{i+1}$ .

We will also denote  $\tau_i \simeq \epsilon_i \circ \alpha_i \simeq v_i \circ \delta_i$ . Notice that  $\alpha_i: A \rightarrow G_i$  is nothing but the whisker map of  $\tau_i: A \rightarrow T_i$  and  $\iota_X: A \rightarrow X$  induced by the right homotopy pullback of Theorem 25.

Theorem 25 allows to give a ‘Whitehead version’ of sectional and relative categories:

**Proposition 26.** *Let  $\iota_X: A \rightarrow X$  be a map of  $\mathcal{M}$ .*

1) *We have  $\text{secat}(\iota_X) \leq n$  if and only if there exists a map  $\rho: X \rightarrow T_n(\iota_X)$  such that  $t_n \circ \rho \simeq \Delta_{n+1}$ .*

2) *We have  $\text{relcat}(\iota_X) \leq n$  if and only if there exists a map  $\rho: X \rightarrow T_n(\iota_X)$  such that  $t_n \circ \rho \simeq \Delta_{n+1}$  and  $\rho \circ \iota_X \simeq \tau_n$ .*

*Proof.* If  $\text{secat}(\iota_X) \leq n$ , we have a map  $\sigma: X \rightarrow G_n$  such that  $g_n \circ \sigma \simeq \text{id}_X$ . Set  $\rho \simeq \epsilon_n \circ \sigma$ . We have  $t_n \circ \rho \simeq t_n \circ \epsilon_n \circ \sigma \simeq \Delta_{n+1} \circ g_n \circ \sigma \simeq \Delta_{n+1}$ . Moreover if  $\text{relcat}(\iota_X) \leq n$ , we have also  $\sigma \circ \iota_X \simeq \alpha_n$ , thus  $\rho \circ \iota_X \simeq \epsilon_n \circ \sigma \circ \iota_X \simeq \epsilon_n \circ \alpha_n \simeq \tau_n$ .

For the reverse direction, assume we have a map  $\rho: X \rightarrow T_n$  such that  $t_n \circ \rho \simeq \Delta_{n+1}$ . The right homotopy pullback of Theorem 25 induces a whisker map  $\sigma: X \rightarrow G_n$  of  $\rho$  and  $\text{id}_X$ , so  $g_n \circ \sigma \simeq \text{id}_X$  and  $\epsilon_n \circ \sigma \simeq \rho$ . If moreover we have  $\rho \circ \iota_X \simeq \tau_n$ , then extend the left homotopy pullback of Theorem 25 to the following

homotopy commutative diagram:

$$\begin{array}{ccccc}
 & & \iota_X & & \\
 A & \xrightarrow{\hat{\sigma}} & P & \xrightarrow{\quad} & X \\
 & \searrow \delta_n & \downarrow \pi & & \downarrow \sigma \\
 & & A & \xrightarrow{\alpha_n} & G_n \\
 & & \downarrow \delta_n & & \downarrow \epsilon_n \\
 & & X^n \times A & \xrightarrow{v_n} & T_n
 \end{array}
 \quad \begin{array}{l} \rho \end{array}$$

where both squares (and the rectangle) are homotopy pullbacks and, as the outer diagram commutes up to homotopy, we have a whisker map  $\hat{\sigma}: A \rightarrow P$  of  $\delta_n$  and  $\iota_X$ ; so we have  $\delta_n \simeq \delta_n \circ \pi \circ \hat{\sigma}$ . As  $\delta_n$  has an obvious (homotopy) retraction, it is a homotopy monomorphism, thus  $\text{id}_A \simeq \pi \circ \hat{\sigma}$ . Finally  $\sigma \circ \iota_X \simeq \alpha_n \circ \pi \circ \hat{\sigma} \simeq \alpha_n$ .  $\square$

**1.6. Change of base.** We return to sectional category and look at its behaviour when the base of the map changes. The precise statement, Proposition 28, is a consequence of the following result:

**Lemma 27.** *Suppose we are given any homotopy commutative diagram in  $\mathcal{M}$ :*

$$\begin{array}{ccc}
 B & \xrightarrow{\kappa_Y} & Y \\
 \zeta \downarrow & & \downarrow f \\
 A & \xrightarrow{\iota_X} & X
 \end{array}$$

*For any  $i \geq 0$ , there is a homotopy commutative diagram in  $\mathcal{M}$ :*

$$\begin{array}{ccc}
 G_i(\kappa_Y) & \xrightarrow{g_i(\kappa_Y)} & Y \\
 \zeta_i \downarrow & & \downarrow f \\
 G_i(\iota_X) & \xrightarrow{g_i(\iota_X)} & X
 \end{array}$$

*Moreover, if the first diagram is a homotopy pullback, the second one is a homotopy pullback as well.*

*Proof.* Use the Join theorem 51 inductively.  $\square$

The three following propositions are straightforward consequences of the previous lemma.

**Proposition 28.** *Suppose we are given any homotopy commutative diagram in  $\mathcal{M}$ :*

$$\begin{array}{ccc}
 B & \xrightarrow{\kappa_Y} & Y \\
 \zeta \downarrow & & \downarrow f \\
 A & \xrightarrow{\iota_X} & X
 \end{array}$$

- 1) *If  $f$  has a homotopy section, then  $\text{secat}(\iota_X) \leq \text{secat}(\kappa_Y)$ .*
- 2) *If the square is a homotopy pullback, then  $\text{secat}(\kappa_Y) \leq \text{secat}(\iota_X)$ .*
- 3) *If  $f$  and  $\zeta$  have homotopy inverses, then  $\text{secat}(\iota_X) = \text{secat}(\kappa_Y)$ .*

**Proposition 29.** *If a map  $\kappa_X: B \rightarrow X$  factors through  $\iota_X: A \rightarrow X$  up to homotopy, then  $\text{secat}(\iota_X) \leq \text{secat}(\kappa_X)$ .*

In particular, if  $\mathcal{M}$  is pointed, for any map  $\iota_X: A \rightarrow X$ ,  $\text{secat}(X, A) \leq \text{cat}(X)$ .

It is of course *not* true that  $\text{relcat}(X, A) \leq \text{cat}(X)$ ; consider  $A \rightarrow *$ , or see Example 21 for instance.

**Proposition 30.** *Let  $\iota_X: A \rightarrow X$  and  $f: Y \rightarrow X$  be maps of  $\mathcal{M}$  and consider the homotopy pullback  $F = Y \times_X A$ . We have  $\text{secat}(Y, F) \leq n$  if and only if  $f$  factors through  $g_n: G_n(\iota_X) \rightarrow X$  up to homotopy.*

This last proposition is an extension of a result of A.S. Schwarz [13].

**Definition 31.** Assume  $\mathcal{M}$  is pointed. Let  $f: Y \rightarrow X$  be a map of  $\mathcal{M}$ . The *category* of  $f$  is the least integer  $n$  such that  $f$  factors through  $g_n: G_n(X) \rightarrow X$  up to homotopy.

The category of  $f$  is denoted by  $\text{cat}(f)$ . By Proposition 30,  $\text{cat}(f) = \text{secat}(Y, F)$  where  $F \rightarrow Y$  is the homotopy fibre of  $f$ .

**1.7. Sectional and relative categories of maps in the Ganea construction.** Consider the Ganea construction of  $\iota_X: A \rightarrow X$ . The following results determine the sectional category and the relative category of the maps  $\alpha_i: A \rightarrow G_i$  and  $\beta_i: F_i \rightarrow G_i$ .

**Theorem 32.** *Let  $\iota_X: A \rightarrow X$  be any map of  $\mathcal{M}$ . Consider the map  $\beta_i: F_i(\iota_X) \rightarrow G_i(\iota_X)$  of the Ganea construction. We have:*

$$\text{relcat}(\beta_i) = \min\{i + 1, \text{relcat}(\iota_X)\} \text{ and } \text{secat}(\beta_i) = \min\{i, \text{secat}(\iota_X)\}.$$

*Proof.* Let  $i \geq \text{secat}(\iota_X)$ . By Theorem 18,  $i + 1 \geq \text{relcat}(\iota_X)$ . Then by Proposition 23 and Corollary 16, we have  $\text{relcat}(\iota_X) = \text{relcat}(\alpha_{i+1}) \leq \text{relcat}(\beta_i) \leq \text{relcat}(\iota_X)$ . On the other hand, since  $F_i = G_i \times_X A$  and  $g_i$  has a homotopy section, Proposition 28 yields  $\text{secat}(\beta_i) = \text{secat}(\iota_X)$ .

Let  $i < \text{secat}(\iota_X) \leq \text{relcat}(\iota_X)$ . Then by Proposition 23 and Corollary 16, we have  $i + 1 = \text{relcat}(\alpha_{i+1}) \leq \text{relcat}(\beta_i)$ . On the other hand, since  $\alpha_i \simeq \beta_i \circ \theta_i$ , by Proposition 29 and Proposition 23, we have  $\text{secat}(\beta_i) \leq \text{secat}(\alpha_i) \leq \text{relcat}(\alpha_i) = i$ . By Theorem 18,  $\text{secat}(\beta_i)$  and  $\text{relcat}(\beta_i)$  can only differ by 1, so  $\text{secat}(\beta_i) = i$  and  $\text{relcat}(\beta_i) = i + 1$ .  $\square$

As a particular case, we have:

**Corollary 33.** *Assume  $\mathcal{M}$  is pointed. Consider the homotopy fibre  $\beta_i: F_i(X) \rightarrow G_i(X)$  of the map  $g_i: G_i(X) \rightarrow X$ . We have*

$$\text{relcat}(\beta_i) = \min\{i + 1, \text{cat}(X)\} \text{ and } \text{cat}(g_i) = \text{secat}(\beta_i) = \min\{i, \text{cat}(X)\}.$$

We are now ready to enhance Proposition 23.

**Corollary 34.** *Let  $\iota_X: A \rightarrow X$  be any map of  $\mathcal{M}$ . Consider the map  $\alpha_i: A \rightarrow G_i(\iota_X)$  of the Ganea construction. We have:*

$$\min\{i, \text{secat}(\iota_X)\} \leq \text{secat}(\alpha_i) \leq \text{relcat}(\alpha_i) = \min\{i, \text{relcat}(\iota_X)\}$$

*Proof.* The last equality is Proposition 23. On the other hand, by Proposition 29 and Theorem 32, we have  $\text{secat}(\alpha_i) \geq \text{secat}(\beta_i) = \min\{i, \text{secat}(\iota_X)\}$ .  $\square$

One might expect that the first inequality would be an equality, but it is not. When  $i > \text{secat}(\iota_X)$  the inequality can be strict. Indeed consider  $\iota_*: A \rightarrow *$ . We have  $\alpha_1: A \rightarrow A \bowtie A$  (join of  $A$  with itself), which is a null map, i.e. it factors through the zero object, so it can not have a homotopy section (unless  $A \simeq *$ ). Thus we have  $\min\{1, \text{secat}(\iota_*)\} = \text{secat}(\iota_*) = 0$ , while  $\text{secat}(\alpha_1) = 1$ .

## 2. COMPLEXITY

In this section,  $\mathcal{M}$  is pointed with  $*$  as zero object.

### 2.1. Complexity.

**Definition 35.** Let  $X$  be any object of  $\mathcal{M}$ .

We define the *complexity* of  $X$  to be the sectional category of the diagonal map  $\Delta: X \rightarrow X \times X$ .

Analogously, we define the *relative* (respectively: *pushout*, and *strong*) *complexity* of  $X$  to be the relative (respectively: pushout, and strong relative) category of the diagonal.

We use the following notations:  $\text{compl}(X) = \text{secat}(\Delta)$ ,  $\text{relcompl}(X) = \text{relcat}(\Delta)$ ,  $\text{Pushcompl}(X) = \text{Pushcat}(\Delta)$ ,  $\text{Compl}(X) = \text{Relcat}(\Delta)$ .

In  $\mathbf{Top}^w$ , the complexity is called *topological complexity* by M. Farber[7] (up to a shift by 1):  $\text{TC}(X) = \text{compl}(X) + 1$ .

Consider the diagonal  $\Delta_{i+1}: X \rightarrow X^{i+1}$  and the maps  $\delta_i(\iota_X): A \rightarrow X^i \times A$  and  $\epsilon_i(\iota_X): G_i(\iota_X) \rightarrow T_i(\iota_X)$  built for any map  $\iota_X: A \rightarrow X$  in Theorem 25.

**Proposition 36.** For any object  $X$  and any map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$ ,

$$\text{cat}(X^i) \leq \text{secat}(\delta_i(\iota_X)) \leq \text{secat}(\epsilon_i(\iota_X)) \leq \text{secat}(\Delta_{i+1}) \leq \text{cat}(X^{i+1}).$$

*Proof.* The last inequality is just Proposition 29.

On the other hand, we have the following homotopy pullbacks:

$$\begin{array}{ccccccc} * & \longrightarrow & A & \longrightarrow & G_i(\iota_X) & \xrightarrow{g_i} & X \\ \downarrow & & \downarrow \delta_i & & \downarrow \epsilon_i & & \downarrow \Delta_{i+1} \\ X^i & \xrightarrow{\text{in}_1} & X^i \times A & \longrightarrow & T_i(\iota_X) & \xrightarrow{t_i} & X^{i+1} \end{array}$$

The two right squares are given by Theorem 25, and the left square is easily obtained by the Prism lemma 46. We deduce the first three inequalities from Proposition 28.  $\square$

This proposition suggests the following definition of complexity of a map:

**Definition 37.** For any map  $\iota_X: A \rightarrow X$ , we define the *complexity* of  $\iota_X$  as the sectional category of  $\delta_1(\iota_X): A \rightarrow X \times A$ .

Recall that  $\delta_1(\iota_X)$  is the whisker map of  $\iota_X$  and  $\text{id}_A$ . We write  $\text{compl}(\iota_X) = \text{compl}(X, A) = \text{secat}(\delta_1(\iota_X))$ .

In particular  $\text{compl}(\text{id}_X) = \text{compl}(X)$ , since  $\delta_1(\text{id}_X) \simeq \Delta$ . On the other hand  $\text{compl}(X, *) = \text{cat}(X)$ ; indeed consider  $\epsilon_1(X): \Sigma\Omega X \rightarrow X \vee X$ , and we see that  $\text{cat}(X) \leq \text{secat}(\delta_1(X)) \leq \text{secat}(\epsilon_1(X)) \leq \text{cat}(X \vee X) = \text{cat}(X)$ .

Proposition 36 gives:

**Corollary 38.** *For any object  $X$  and any map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$ ,*

$$\text{cat}(X) \leq \text{compl}(\iota_X) \leq \text{compl}(X) \leq \text{cat}(X \times X).$$

**Example 39.** M. Farber [7] has shown that the complexity of a sphere is 1 if the dimension is odd and 2 if the dimension is even.

Consider the Hopf fibration  $S^7 \rightarrow S^4$  and factor by the action of  $S^1$  on  $S^7$  to get  $\iota: \mathbb{C}P^3 \rightarrow S^4$ . Let  $u$  be a generator of the rational cohomology  $H^*(S^4)$  and  $v$  be a generator of  $H^*(\mathbb{C}P^3)$ . Define  $a = 1 \otimes v^2 - u \otimes 1$  and  $b = u \otimes v^2$  in  $H^*(S^4) \otimes H^*(\mathbb{C}P^3)$ . The map  $\delta_1: \mathbb{C}P^3 \rightarrow S^4 \times \mathbb{C}P^3$  induces  $\delta_1^*: H^*(S^4) \otimes H^*(\mathbb{C}P^3) \rightarrow H^*(\mathbb{C}P^3)$ . We have  $\delta_1^*(a) = 1 \cdot v^2 - v^2 \cdot 1 = 0$  and  $a^2 = -2b \neq 0$ . By A.S. Schwarz [13], Theorem 4, this means that  $2 \leq \text{secat}(\delta_1) = \text{compl}(\iota)$ . But by Corollary 38, we also know that  $\text{compl}(\iota) \leq \text{cat}(S^4 \times S^4) = 2$ . So  $\text{compl}(\iota) = 2$ .

In contrast, from the fact that  $\text{compl}(S^{2n+1}) = 1$  and Corollary 38, we see that  $\text{compl}(\iota') = 1$  for any map  $\iota': A \rightarrow S^{2n+1}$ .

On the other hand, by Theorem 18, we know that all the variants of complexity can only differ by 1:

**Proposition 40.** *For any object  $X$  of  $\mathcal{M}$ ,*

$$\text{compl}(X) \leq \text{relcompl}(X) \leq \text{Pushcompl}(X) \leq \text{Compl}(X) \leq \text{compl}(X) + 1.$$

Observe the following other lower bound of the strong complexity:

**Proposition 41.** *For any object  $X$  and any map  $\iota_X: A \rightarrow X$  of  $\mathcal{M}$ ,*

$$\text{Cat}(X^i) \leq \text{Relcat}(\delta_i(\iota_X)) \leq \text{Relcat}(\epsilon_i(\iota_X)) \leq \text{Relcat}(\Delta_{i+1}).$$

In particular,  $\text{Cat}(X) \leq \text{Compl}(X)$ .

*Proof.* Use Lemma 14 with the same homotopy pullbacks as in Proposition 36.  $\square$

**2.2. Complexity of a suspension.** Let us consider the *pinch* map  $p: \Sigma X \rightarrow \Sigma X \vee \Sigma X$  which is the whisker map induced by the top homotopy pushout in the following homotopy commutative diagram:

$$\begin{array}{ccccc} & X & \xrightarrow{\quad} & * & \\ & \swarrow & & \swarrow & \searrow \\ * & \xrightarrow{\quad} & \Sigma X & \xrightarrow{\quad} & \Sigma X \\ \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{\quad} & \Sigma X & \xrightarrow{\quad} & \Sigma X \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\quad} & \Sigma X \vee \Sigma X & \xrightarrow{\quad} & \Sigma X \times \Sigma X \\ \text{in}_1 \uparrow & & \text{in}_2 \uparrow & & \uparrow \Delta \end{array}$$

(Note: The diagram is a homotopy commutative diagram with nodes  $X, *, \Sigma X, \Sigma X \vee \Sigma X, \Sigma X \times \Sigma X$  and arrows including  $p, \text{in}_1, \text{in}_2, \Delta$ .)

where  $\text{in}_1$  and  $\text{in}_2$  are the obvious maps. The outside part of the diagram is described in Example 47 in the Appendix, and it is extended to the whole homotopy commutative diagram using Lemma 49.

**Lemma 42.** *We have  $\text{Relcat}(p) \leq 1$ .*

*Proof.* Observe that the faces of the inside cube in the preceeding diagram are homotopy pushouts (use the Prism lemma 46 to show that the front and right squares are indeed homotopy pushouts). So  $p$  appears as the homotopy cofibre of  $X \rightarrow \Sigma X$  and Proposition 9 gives the result.  $\square$



Actually we have the following explicit homotopy commutative diagram with a homotopy pushout, where  $\text{in}_1$  and  $\text{pr}_1$  are the obvious maps:

$$\begin{array}{ccccc} \Sigma X & \xrightarrow{\text{in}_1} & \Sigma X \vee X & \xrightarrow{\text{pr}_1} & \Sigma X \\ & \searrow & \downarrow \text{pr}_1 & & \downarrow p \\ & & \Sigma X & \longrightarrow & \Sigma X \vee \Sigma X \end{array}$$

and Definition 8 yields  $\text{Relcat}(p) \leq 1$  directly.

**Theorem 43.** *Let  $X$  be any object of  $\mathcal{M}$ . We have*

$$\text{Compl}(\Sigma X) \leq 2.$$

*Proof.* From Proposition 52 we have a homotopy cofibration sequence:

$$X \bowtie X \xrightarrow{u} \Sigma X \vee \Sigma X \xrightarrow{t_1} \Sigma X \times \Sigma X$$

So Proposition 50 gives the following homotopy pushout:

$$\begin{array}{ccc} (\Sigma X \vee \Sigma X) \vee (X \bowtie X) & \xrightarrow{(\text{id}, u)} & \Sigma X \vee \Sigma X \\ \text{pr}_1 \downarrow & & \downarrow t_1 \\ \Sigma X \vee \Sigma X & \xrightarrow{t_1} & \Sigma X \times \Sigma X \end{array}$$

We now extend this square to the following homotopy commutative diagram:

$$\begin{array}{ccccc} \Sigma X \vee (X \bowtie X) & \xrightarrow{p \vee \text{id}} & (\Sigma X \vee \Sigma X) \vee (X \bowtie X) & \xrightarrow{(\text{id}, u)} & \Sigma X \vee \Sigma X \\ \text{pr}_1 \downarrow & & \text{pr}_1 \downarrow & & \downarrow t_1 \\ \Sigma X & \xrightarrow{p} & \Sigma X \vee \Sigma X & \xrightarrow{t_1} & \Sigma X \times \Sigma X \end{array}$$

where  $p \vee \text{id}$  is the whisker map of  $\text{in}_1 \circ p$  and  $\text{in}_2$ . The two squares are homotopy pushouts, so the outside rectangle is a homotopy pushout, too. Recall that  $t_1 \circ p \simeq \Delta$ . We now obtain a homotopy commutative diagram with a homotopy pushout:

$$\begin{array}{ccccc} \Sigma X & \xrightarrow{\text{in}_1} & \Sigma X \vee (X \bowtie X) & \xrightarrow{\text{pr}_1} & \Sigma X \\ & \searrow p & \downarrow (p, u) & & \downarrow \Delta \\ & & \Sigma X \vee \Sigma X & \xrightarrow{t_1} & \Sigma X \times \Sigma X \end{array}$$

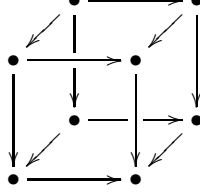
Finally Lemma 42 and Lemma 17 yield the desired result.  $\square$

**Example 44.** As mentioned before, for any  $n > 0$ ,  $\text{compl}(S^{2n}) = 2$ . Therefore, by Proposition 40 and Theorem 43, we see that  $\text{Compl}(S^{2n}) = 2$ .

## APPENDIX A. TOOLBOX

All constructions made in this paper can be achieved in a closed model category  $\mathcal{M}$  satisfying the following additional axiom:

**Axiom 45** (Cube axiom). For any homotopy commutative diagram in  $\mathcal{M}$ :

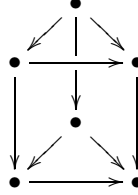


if the bottom face is a homotopy pushout and the four vertical faces are homotopy pullbacks, then the top face is a homotopy pushout.

Of course this axiom is satisfied in the category of topological spaces; see [12], Theorem 25.

The following lemma is used very often, sometimes implicitly:

**Lemma 46** (Prism lemma). *Suppose given any homotopy commutative diagram in  $\mathcal{M}$ :*

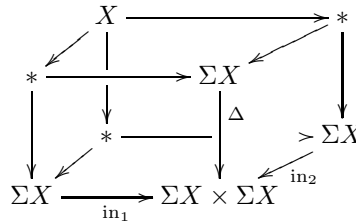


- 1) *Suppose the left square is a homotopy pushout. Then the front square is a homotopy pushout if and only if the right square is a homotopy pushout.*
- 2) *Suppose the front square is a homotopy pullback. Then the left square is a homotopy pullback if and only if the right square is a homotopy pullback.*

By a ‘homotopy commutative diagram’ we mean not only a set of maps, but also a choice of homotopies between (composites of) maps with same source and target, and which are ‘compatible’ with each other; see [12] or [5] for details.

In particular, a ‘homotopy commutative square’ involves the choice of one homotopy.

**Example 47.** Consider the following diagram:



in the category of pointed topological spaces, where  $\text{in}_1$  and  $\text{in}_2$  are the obvious maps and  $\Sigma X$  is the reduced suspension of  $X$ , with the homotopy  $H: X \times I \rightarrow \Sigma X: (x, t) \mapsto [x, t]$  attached to the top, left and back squares, and (of course) the static homotopy attached to the front, right and bottom squares. The left, back

and bottom squares yield a homotopy:

$$K: X \times I \rightarrow \Sigma X \times \Sigma X: (x, t) \mapsto \begin{cases} ([x, 2t], *) & \text{if } t \leq \frac{1}{2} \\ (*, [x, 2t - 1]) & \text{if } t \geq \frac{1}{2} \end{cases}$$

The top, front and right squares yield a homotopy:

$$L: X \times I \rightarrow \Sigma X \times \Sigma X: (x, t) \mapsto ([x, t], [x, t])$$

The homotopies are compatible (we write  $K \sim L$ ) thanks to the following higher homotopy:

$$M: X \times I \times I \rightarrow \Sigma X \times \Sigma X: (x, t, s) \mapsto \begin{cases} ([x, \frac{2t}{s+1}], *) & \text{if } t \leq \frac{1-s}{2} \\ ([x, \frac{2t}{s+1}], [x, \frac{2t+s-1}{s+1}]) & \text{if } \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ (*, [x, \frac{2t+s-1}{s+1}]) & \text{if } t \geq \frac{1+s}{2} \end{cases}$$

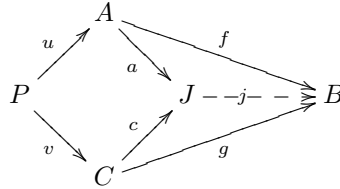
We don't write the homotopies explicitly in the paper because in most cases, all we have to know is that they are there! Many diagrams are built using homotopy pushouts and/or homotopy pullbacks constructions, and in this case the homotopies are well defined (up to equivalences) and the diagrams are naturally homotopy commutative. Lemma 49 below illustrates this fact. However, it is important to keep in mind that all these homotopies are still there and are well defined (up to equivalences).

We write  $f \simeq g$  when the map  $f$  is homotopic to  $g$ .

We denote by  $U \vee_V W$  the homotopy pushout of maps  $U \longleftarrow V \longrightarrow W$  and by  $A \times_B C$  the homotopy pullback of maps  $A \longrightarrow B \longleftarrow C$ .

When the category  $\mathcal{M}$  is pointed, i.e. it has a *zero* object  $*$  both initial and terminal, we omit to write the subscript on  $\vee$  or  $\times$  if it is the zero object. The homotopy pushout  $U \vee_V *$  is denoted by  $U/V$ , the map  $U \rightarrow U/V$  (or  $U/V$  itself) is called the *homotopy cofibre* of  $V \rightarrow U$  and the sequence  $V \longrightarrow U \longrightarrow U/V$  is called a *homotopy cofibration*. Let  $F = E \times_B *$ ; the map  $F \rightarrow E$  (or  $F$  itself) is called *homotopy fibre* of  $E \rightarrow B$ . Finally, we write  $\Sigma X = * \vee_X *$ , called *suspension* of  $X$ , and  $\Omega X = * \times_X *$ .

Consider two homotopy commutative squares in  $\mathcal{M}$ , where the inside square is a homotopy pushout:



These two squares and the two attached homotopies  $H: a \circ u \simeq c \circ v$  and  $K: f \circ u \simeq g \circ v$  induce a map  $j: J \rightarrow B$ , called the *whisker* map, together with homotopies of the triangles  $M: f \simeq j \circ a$  and  $N: j \circ c \simeq g$  making the whole diagram homotopy commutative, that is  $M \circ (u \times \text{id}_I) + j \circ H + N \circ (v \times \text{id}_I) \sim K$ . The map  $j$  is ‘universal’ in the following sense: If there is another map  $j'$  and (other) homotopies  $M': f \simeq j' \circ a$  and  $N': j' \circ c \simeq g$  of the triangles making the whole diagram homotopy commutative, then there is a homotopy  $L: j \simeq j'$ , and the diagram with all maps, including  $j$  and  $j'$ , and all homotopies, is homotopy commutative, that is  $M + L \circ (a \times \text{id}_I) \sim M'$  and  $L \circ (c \times \text{id}_I) + N' \sim N$ . See [12], Theorem 11.

The whisker map  $j$  is sometimes denoted by  $(f, g)$ .

Despite this notation, we emphasize that  $j$  is determined not only by the outside square but also by the attached homotopy. A different choice of homotopy lead to a different induced map. In other words, the whisker map is unique (up to homotopy) *once* the homotopy is fixed. For instance, in the category of pointed topological spaces, if  $A$  and  $C$  are the one point set  $\{*\}$ , then  $J$  is the reduced suspension  $\Sigma P$  and the homotopy attached to the homotopy pushout is  $H : P \times I \rightarrow \Sigma P : (x, t) \mapsto [x, t]$ . Let  $B = J = \Sigma P$ . If the homotopy attached to the outside square is  $H$ , i.e. the same as the homotopy attached to the inside square, then  $j$  is the identity. But if the homotopy attached to the outside square is the static one, then  $j$  is the null map.

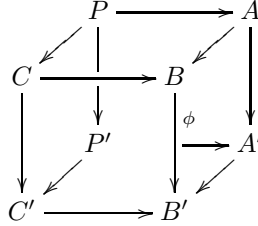
There is a ‘dual’ notion of whisker map for homotopy pullbacks.

**Definition 48** (Join). If in the above diagram the outside square is a homotopy pullback, then the whisker map  $j$  induced by the homotopy pushout (or  $J$  itself) is called the *join* of  $f$  and  $g$ .

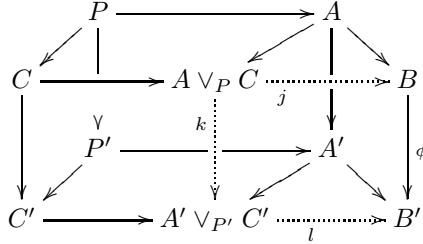
We denote the join  $J$  by  $A \bowtie_B C$ . We omit the subscript on  $\bowtie$  if it is the zero object.

Ganea and Whitehead constructions are particular cases of join constructions.

**Lemma 49** (Whisker maps inside a cube). *Given a homotopy commutative cube:*

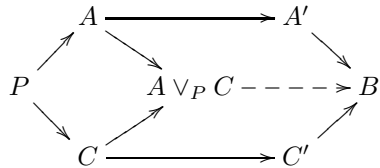


*it can be extended to a homotopy commutative diagram:*



where  $j$  and  $k$  are the whisker maps induced by the top homotopy pushout and  $l$  is the whisker map induced by the bottom homotopy pushout.

*Proof.* The top part, the inside cube, and the bottom part of the diagram are homotopy commutative by construction of the whisker maps  $j$ ,  $k$  and  $l$  respectively. Moreover both  $l \circ k$  and  $\phi \circ j$  make the following diagram homotopy commutative:



So by the universal property of the whisker map,  $l \circ k \simeq \phi \circ j$  and the whole diagram is homotopy commutative.  $\square$

**Proposition 50.** *Let  $\mathcal{M}$  be pointed. Let be given any homotopy cofibration sequence  $Y \xrightarrow{f} A \xrightarrow{g} X$ . There is a homotopy pushout:*

$$\begin{array}{ccc} A \vee Y & \xrightarrow{(\text{id}_A, f)} & A \\ \text{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{g} & X \end{array}$$

where  $(\text{id}_A, f)$  is the whisker map of  $\text{id}_A$  and  $f$ .

*Proof.* Apply Lemma 49 to the outside part of the following diagram:

$$\begin{array}{ccccc} & & * & \xrightarrow{\quad} & Y \\ & \swarrow & \downarrow & \nearrow \text{in}_2 & \downarrow f \\ A & \xrightarrow{\text{in}_1} & A \vee Y & \xrightarrow{(\text{id}, f)} & A \\ \parallel & \searrow \gamma & \downarrow \text{pr}_1 & & \downarrow g \\ A & \xrightarrow{\quad} & * & \xrightarrow{\quad} & X \\ & \swarrow & \downarrow & \nwarrow & \\ & A & \xrightarrow{g} & X & \end{array}$$

We get the whole diagram homotopy commutative. Moreover, using the Prism lemma 46 we get that all squares, except the back one and the leftern front one, are homotopy pushouts.  $\square$

**Theorem 51** (Join theorem). *Suppose we have two homotopy commutative squares:*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longleftarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longleftarrow & C' \end{array}$$

Then there is a homotopy commutative diagram:

$$\begin{array}{ccccc} A & \longrightarrow & A \bowtie_B C & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & A' \bowtie_{B'} C' & \longrightarrow & B' \end{array}$$

Moreover, if the two squares in the first diagram are homotopy pullbacks, then the two squares in the second diagram are homotopy pullbacks as well.

*Proof.* We construct the following homotopy commutative diagram

$$\begin{array}{ccccc} A \times_B C & \xrightarrow{\quad} & A & & \\ \swarrow & \downarrow & \swarrow & \downarrow & \searrow \\ C & \xrightarrow{\quad} & A \bowtie_B C & \xrightarrow{\quad} & B \\ \downarrow & \searrow & \downarrow & \searrow & \downarrow \\ & A' \times_{B'} C' & \xrightarrow{\quad} & A' & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ C' & \xrightarrow{\quad} & A' \bowtie_{B'} C' & \xrightarrow{\quad} & B' \end{array}$$

where  $A \times_B C \dashrightarrow A' \times_{B'} C'$  is the whisker map induced by the bottom homotopy pullback and  $A \bowtie_B C \dashrightarrow A' \bowtie_{B'} C'$  is the whisker map induced by the top homotopy pushout. The rightern front vertical square is homotopy commutative by Lemma 49.

If the two squares we start with are homotopy pullbacks, then the Prism lemma and the Cube axiom imply that all vertical faces of the above diagram are also homotopy pullbacks.  $\square$

**Proposition 52.** *Assume that  $\mathcal{M}$  is pointed, and let  $X \rightarrow A \rightarrow A'$  and  $Y \rightarrow B \rightarrow B'$  be two homotopy cofibration sequences. Consider the following join construction:*

$$\begin{array}{ccccc}
 & & A' \times B & & \\
 & \nearrow & & \searrow & \\
 A \times B & & & & J \dashrightarrow A' \times B' \\
 & \searrow & & \nearrow & \\
 & & A \times B' & & 
 \end{array}$$

Then, there is a homotopy cofibration sequence:

$$X \bowtie Y \longrightarrow J \xrightarrow{j} A' \times B'$$

This proposition relies on the Cube axiom; see [11] or [2], Proposition B.35.

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